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Canonical orthonormal basis for SU(3) ⊃ SO(3). I: Construction of the basis

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Abstract. A canonical orthonormal basis is given for generic representations of the group chain SU(3) ⊃ SO(3).

1. Introduction

In a recent series of papers, Deenen and Quesne (1983) and Quesne (1984a, b) succeeded in constructing a generally non-orthonormal basis for the group chain U(n) ⊃ O(n) for which the internal labelling problem is resolved by sound group theoretical techniques.

Consider, for example, the case $n=3$. The idea is to classify states in the fundamental unirreps of Sp(6, $\mathfrak{R}$) by their transformation properties under Sp(2, $\mathfrak{R}$) × O(3) and its subgroups. Due to the complementarity of Sp(2, $\mathfrak{R}$) and O(3) on this space (Chacón 1969, Moshinsky and Quesne 1970, 1971), states that belong to a unirrep $\langle\frac{3}{2}(L\varepsilon)\rangle$ of Sp(2, $\mathfrak{R}$) simultaneously belong to a unirrep $[L\varepsilon]$ of O(3) (the notation will be explained below). Likewise, states that belong to a unirrep $\{\frac{3}{2}(h_1h_2)\}$ of U(2) ⊂ Sp(2, $\mathfrak{R}$) simultaneously belong to a unirrep $\{h_1h_2 0\}$ of U(3). Thus the Sp(2, $\mathfrak{R}$) and O(3) groups share their labels and the U(2) and U(3) groups share theirs. We then obtain a complete classification of states by

$$\begin{array}{ccccccc} & \mathrm{Sp}(2,\mathfrak{R}) & \supset & \mathrm{U}(2) & \supset & \mathrm{U}(1) & \\ M & \langle\frac{3}{2}(L\varepsilon)\rangle & (\delta_1\delta_2) & \{\frac{3}{2}(h_1h_2)\} & & \mu & \\ \mathrm{O}(2)\subset & \mathrm{O}(3) & \subset & \mathrm{U}(3) & & & . \end{array}$$

The vital property of this classification, observed by Deenen and Quesne, is that the extra labels for the U(3)↓O(3) reduction can be identified with the missing labels for the Sp(2, $\mathfrak{R}$)↓U(2) reduction.

More generally, in order to construct a U(n) ⊃ O(n) basis for the d-rowed representations of U(n), one similarly has to classify states in the fundamental unirreps of Sp(nd, $\mathfrak{R}$) by their transformation properties under Sp(d, $\mathfrak{R}$) × O(n) and its subgroups, thereby identifying the U(n)↓O(n) reduction with the Sp(d, $\mathfrak{R}$)↓U(d) reduction. A thorough study of the branching rules relevant to these reductions has recently been given by Rowe *et al* (1985b).

In this paper, we take the analysis of Deenen and Quesne one step further to obtain canonical orthonormal bases for SU(3) ⊃ SO(3). Although our premises are very similar to those of Deenen and Quesne (1983) and Quesne (1984a, b), our construction will be seen to be different. The group theoretical prescription for the construction is unambiguous and readily extendable to the more general case of constructing orthonormal bases for d-rowed representations of $\mathrm{SU}(n) \supset \mathrm{SO}(n)$. Using recent advances in the representation theory of the discrete series of the non-compact symplectic groups $\mathrm{Sp}(d, \mathfrak{R})$ (Rowe 1984, Rowe *et al* 1985a, b), we will make use of the fact that orthonormal bases for $\mathrm{Sp}(d, \mathfrak{R}) \supset \mathrm{U}(d)$ can be explicitly constructed in terms of the reduction of the outer product of $\mathrm{U}(d)$ unirreps which have been analysed in depth by Biedenharn and Louck and their collaborators (1972 and references therein). Thus, given a canonical reduction of the $\mathrm{U}(d)$ outer products (which, in view of results given in part III of this series, appears feasible), we obtain a canonical $\mathrm{U}(n) \supset \mathrm{O}(n)$ basis for the d-rowed $\mathrm{U}(n)$ representations. For $d = 2$, there is no outer multiplicity and we obtain immediately a canonical $\mathrm{SU}(3) \supset \mathrm{SO}(3)$ basis.

Lest it appears laboured to use an $\mathrm{Sp}(d, \mathfrak{R}) \supset \mathrm{U}(d)$ reduction to solve a $\mathrm{U}(n) \supset \mathrm{O}(n)$ problem, it should be made clear that it is, in fact, simply an explicit implementation of the structure underlying Littlewood's theorem (cf equation (2.1)) which gives the $\mathrm{U}(n) \supset \mathrm{O}(n)$ branching rules for d-rowed representations of $\mathrm{U}(n)$ in terms of outer (Kronecker) products of $\mathrm{U}(d)$ unirreps.

2. The state labelling problem for $\mathrm{U}(n) \supset \mathrm{O}(n)$ and $\mathrm{Sp}(d, \mathfrak{R}) \supset \mathrm{U}(d)$

Consider a unirrep of $\mathrm{U}(n)$ with character $\{h\}$ where

$$h = (h_1 h_2 \ldots h_d)$$

is a standard partition of integers having $d < n$.

An $\mathrm{O}(n)$ character is denoted by $[\lambda]$ where λ is a partition of integers having not more than ν parts for $n = 2\nu$ or $2\nu + 1$. The reduction of a $\mathrm{U}(n)$ unirrep under the restriction $\mathrm{U}(n)\downarrow\mathrm{O}(n)$ is given by Littlewood's branching theorem which states that if the Kronecker product of two $\mathrm{U}(n)$ unirreps $\{\lambda\}$ and $\{\mu\}$ contains the unirrep $\{h\}$ a number of times denoted by $g_{\{\lambda\}\{\mu\}\{h\}}$, then, under the restriction $\mathrm{U}(n)\downarrow\mathrm{O}(n)$,

$$\{h\}\downarrow\sum_{\lambda}\sum_{\delta\in D} g_{\{\lambda\}\{\delta\}\{h\}}[\lambda] \tag{2.1}$$

where D is the set of partitions (Black *et al* 1983) of even integer parts.

Note, however, that if $2d > n$, non-standard $\mathrm{O}(n)$ labels occur in this reduction which may be converted to standard labels using Newell's (1951) or King's (1975) modification rules. For example, if $n = 2\nu + 1$, $d = \nu + 1$, $\nu \geqslant 1$, one encounters non-standard $\mathrm{O}(n)$ labels of the type $[\lambda\, 1]$ where λ is a partition of ν integer parts. The modification rules give $[\lambda\, 1] \equiv [\lambda]^*$ where

$$\begin{aligned} &[\lambda]^* = \varepsilon[\lambda] \\ &\varepsilon(g) = \det(g) \qquad g \in \mathrm{O}(N) \end{aligned} \tag{2.2}$$

and the branching rule becomes

$$\{h\}\downarrow\sum_{\lambda}\sum_{\delta\in D} g_{\{\lambda\}\{\delta\}\{h\}}[\lambda] + \sum_{\lambda}\sum_{\delta\in D} g_{\{\lambda\, 1\}\{\delta\}\{h\}}[\lambda]^* \tag{2.3}$$

where now λ, for the first term, is restricted to partitions of no more than ν parts and, for the second term, to strictly ν parts. For $n = 3$ and $d = 2$, this is the only modification needed. However, for the general case, other modifications will be required.

For $d < n$, the U(n) unirrep $\{h\}$ restricts to a unirrep $\{h\}$ of SU(n). Also, for $n = 2\nu + 1$, the distinct O(n) unirreps $[\lambda]$ and $[\lambda]^*$ remain irreducible on restriction to SO(n) but, since $\varepsilon(g) = 1$ for SO(n), they become equivalent.

For example, the multiplicity of an SO(3) unirrep $[L]$ contained in an SU(3) unirrep $\{h_1 h_2\}$ is equal to

$$\begin{aligned} &\sum_{\delta \in D} g_{\{L0\}\{\delta_1\delta_2\}\{h_1h_2\}} && h_1 + h_2 - L \text{ even} \\ &\sum_{\delta \in D} g_{\{L_1 1\}\{\delta_1\delta_2\}\{h_1h_2\}} && h_1 + h_2 - L \text{ odd}. \end{aligned} \tag{2.4}$$

The construction of a basis of states $|\{h\}\alpha[\lambda]\xi\rangle$, where α is a multiplicity index for the U(n)$\downarrow$O(n) reduction and ξ indexes a basis for the O(n) irrep $[\lambda]$, is greatly facilitated by the use of the Chacón-Moshinsky-Quesne complementarity theorem. The theorem states that, in a Bargmann space of $n \times d$ complex variables or equivalently, in the space of the simple nd-dimensional harmonic oscillator, states that belong to a unirrep $\langle(n/2)(\lambda)\rangle$ of Sp($d, \mathfrak{R}$) also belong to a unirrep $[\lambda]$ of O(n), where we define

$$(n/2)(\lambda) \equiv (\lambda_1 + n/2, \lambda_2 + n/2, \ldots, \lambda_d + n/2). \tag{2.5}$$

Likewise, states that belong to a unirrep $\{(n/2)(h)\}$ of U(d) also belong to a unirrep $\{h\}$ of U(n).

Note that the same non-standard O(n) labels $[\lambda]$ arise in the complementarity relationship as occur in Littlewood's branching theorem, e.g. for $d = 2$ and $n = 3$, appropriate for the SU(3)$\downarrow$SO(3) reduction, the Sp($2, \mathfrak{R}$) unirrep $\langle\frac{3}{2}(L0)\rangle$ is complementary to the O(3) unirrep $[L]$ and the $\langle\frac{3}{2}(L1)\rangle$ unirrep for $L > 0$ is complementary to $[L1] \equiv [L]^*$ (Rowe *et al* 1985b).

As a consequence of the complementarity theorem, U(d)$\times$O(n) highest weight states for the chain

$$\begin{bmatrix} \mathrm{Sp}(d, \mathfrak{R}) & \supset & \mathrm{U}(d) \\ \langle(n/2)(\lambda)\rangle & \alpha & \{(n/2)(h)\} \end{bmatrix} \times \begin{matrix} \mathrm{O}(n) \\ [\lambda] \end{matrix} \tag{2.6}$$

can be identified with

$$\begin{bmatrix} \mathrm{U}(n) \supset \mathrm{O}(n) \\ \{h\}\alpha[\lambda] \end{bmatrix} \times \begin{matrix} \mathrm{U}(d) \\ \{(n/2)(h)\} \end{matrix} \tag{2.7}$$

basis states. The state labelling problem for the chain U(n)$\supset$O(n) is thus identified with the Sp($d, \mathfrak{R}$)$\supset$U(d) labelling problem.

The algebras of relevance for the construction of an SU(3)$\supset$SO(3) basis are then the sp($2, \mathfrak{R}$) and the u(3) algebras realised here in terms of the same six complex (Bargmann) variables ($g_{\alpha i}$; $\alpha = 1, 2$, $i = 1, 3$). The sp($2, \mathfrak{R}$) algebra is spanned by the 10 operators

$$\begin{aligned} \mathscr{A}_{\alpha\beta} &= g_\alpha \cdot g_\beta = g_{\alpha i} g_{\beta i} \\ \mathscr{B}_{\alpha\beta} &= \nabla_{\alpha\beta} = \frac{\partial^2}{\partial g_{\alpha i} \partial g_{\beta i}} \\ \mathscr{C}_{\alpha\beta} &= g_{\alpha i} \frac{\partial}{\partial g_{\beta i}} + \tfrac{3}{2}\delta_{\alpha\beta} \end{aligned} \tag{2.8}$$

(with summation over repeated indices) while the u(3) algebra is spanned by

$$C_{ij} = g_{\alpha i}\frac{\partial}{\partial g_{\alpha j}} \tag{2.9}$$

with its so(3) subalgebra given by

$$L_i = -i\varepsilon_{ijk}C_{jk}. \tag{2.10}$$

3. Orthonormal basis for Sp(2, $\mathfrak{R}$) $\supset$ U(2)

Recently, Rowe (1984) has developed a coherent state theory for the non-compact symplectic group Sp($d, \mathfrak{R}$). His work makes use of an implicitly defined canonical orthonormal basis for a given unirrep of this group. It was also shown how to calculate reduced matrix elements of the symplectic algebra in this basis. Since we are interested in the explicit construction of an orthonormal Sp(2, $\mathfrak{R}$) basis, it is useful to recall his results in a format applicable to the present case.

A unirrep of the algebra Sp(2, $\mathfrak{R}$) is identified by the quantum labels of its highest weight state, i.e. a state $|\langle(n/2)(\lambda_1\lambda_2)\rangle\mathrm{hw}\rangle$ such that

$$\begin{aligned}&\mathscr{B}_{\alpha\beta}|\langle(n/2)(\lambda_1\lambda_2)\rangle\mathrm{hw}\rangle=0\\&\mathscr{C}_{\alpha\beta}|\langle(n/2)(\lambda_1\lambda_2)\rangle\mathrm{hw}\rangle=0\qquad\alpha<\beta\\&\mathscr{C}_{\alpha\alpha}|\langle(n/2)(\lambda_1\lambda_2)\rangle\mathrm{hw}\rangle=((n/2)+\lambda_\alpha)|\langle(n/2)(\lambda_1\lambda_2)\rangle\mathrm{hw}\rangle.\end{aligned} \tag{3.1}$$

Others states of the unirrep are generated by the multiple action of the lowering operators $\mathscr{A}$ on the highest weight state. Note that we consider here the generic case. Restriction to

$$\langle(n/2)(\lambda_1\lambda_2)\rangle=\langle\tfrac{3}{2}(L\varepsilon)\rangle$$

$\varepsilon=0$ or 1, $L\geqslant\varepsilon$, will be made in subsequent sections.

The lowering operators $\mathscr{A}_{\alpha\beta}$ are the components of a U(2) {20} tensor. A product of k such tensors can be coupled to an irreducible U(2) tensor $Z^{\{\delta\}}$ of rank $\{\delta\}=\{\delta_1\delta_2\}$ where $\delta_1\geqslant\delta_2$ are even integers and $\delta_1+\delta_2=2k$:

$$Z^{\{\delta\}}(\mathscr{A})=[\underbrace{\mathscr{A}\times\mathscr{A}\times\ldots\times\mathscr{A}}_{k\text{ products}}]^{\{\delta\}}. \tag{3.2}$$

Since the {20} tensor $\mathscr{A}$ is uniquely defined up to an amplitude factor, it follows that $Z^{\{\delta\}}(\mathscr{A})$ is also unique up to a k-dependent amplitude and an unimportant phase.

The tensor $Z^{\{\delta\}}$ can also be labelled by

$$\begin{aligned}k&=\tfrac{1}{2}(\delta_1+\delta_2)\\I&=\tfrac{1}{2}(\delta_1-\delta_2)\end{aligned} \tag{3.3}$$

where I is the familiar SU(2) angular momentum. Thus the components of $Z^{\{\delta\}}$ can be indexed by the z component of I given by $m=-I,\ldots+I$.

A non-orthonormal Sp(2, $\mathfrak{R}$) $\supset$ U(2) basis is now given by the coupled states $|\Psi(\langle(n/2)(\lambda)\rangle\{\delta\}\{\omega\}M)\rangle=[Z^{\{\delta\}}(\mathscr{A})|\langle(n/2)(\lambda)\rangle\rangle]^{\{\omega\}}_M$

$$=\sum_{mm'}\langle I\Lambda mm'|JM\rangle Z^{\{\delta\}}_m(\mathscr{A})|\langle(n/2)(\lambda)\rangle m'\rangle \tag{3.4}$$

where

$$\Lambda = \tfrac{1}{2}(\lambda_1 - \lambda_2)$$
$$J = \tfrac{1}{2}(\omega_1 - \omega_2).$$

Thus for $\mathrm{Sp}(2, \mathfrak{R}) \supset \mathrm{U}(2)$, $\{\delta\}$ provides a complete set of missing labels.

The $\mathrm{Sp}(2, \mathfrak{R}) \supset \mathrm{U}(2)$ states for a given unirrep $\langle(n/2)(\lambda)\rangle$ are easily determined by observing that k, I and J run over the ranges

$$\begin{aligned} k &= 0, 1, 2, \ldots, \infty \\ I &= k, k-2, \ldots, 1 \text{ or } 0 \\ J &= |\Lambda - I|, \ldots, \Lambda + I. \end{aligned} \tag{3.5}$$

The corresponding state labels are determined from (3.4) and (3.5) together with the relationship

$$\omega_1 + \omega_2 = \lambda_1 + \lambda_2 + n + 2k. \tag{3.6}$$

One obtains the sequence of states with

$$\begin{array}{lll} k=0 & I=0 & J=\Lambda \\ k=1 & I=1 & J=\Lambda,\ \Lambda\pm 1 \\ k=2 & I=2 & J=\Lambda,\ \Lambda\pm 1,\ \Lambda\pm 2 \\ & I=0 & J=\Lambda. \\ & \vdots & \end{array}$$

Thus the first multiplicity occurs when $k = 2$ and $J = \Lambda$. Such a case will be studied in § 5.

Although the states defined by equation (3.4) are not orthogonal they can be transformed into an orthonormal basis denoted

$$|\langle(n/2)(\lambda)\rangle\{\delta\}\{\omega\}M\rangle \tag{3.7}$$

by means of a simple Hermitian transformation (Rowe 1969)

$$|\langle(n/2)(\lambda)\rangle\{\delta\}\{\omega\}M\rangle = \sum_{\delta'} K^{-1}_{\delta\delta'}[Z^{\{\delta'\}}(\mathscr{A})|\langle(n/2)(\lambda)\rangle\rangle]^{\{\omega\}}_{M} \tag{3.8}$$

where $K = K^{+}$ is the Hermitian square root of the matrix consisting of the overlaps between members of the non-orthonormal basis (3.4)

$$K^2_{ij}((n/2)(\lambda);\omega) = \langle\Psi_{i\omega}|\Psi_{j\omega}\rangle \tag{3.9}$$

and where i and j stand for different (δ) labels. The K^2 matrices are easily calculated algebraically following methods given in the appendix. It should be observed that, since K does not mix states of different k, the arbitrary k-dependent amplitude factors in $Z^{\{\delta\}}(\mathscr{A})$ cancel in equation (3.8) giving basis states that are uniquely determined, up to phase, by their group theoretically defined labels.

The above orthonormalisation procedure is both simple and elegant and has the very distinct advantage over the Gram–Schmidt procedure, for example, that it is independent of any arbitrary choices such as the ordering of the initial non-orthogonal basis. Furthermore, the K matrices appear very naturally in the coherent state theory of the non-compact symplectic groups (see appendix and Rowe 1984) and other compact and non-compact groups (Hecht and Elliott 1985, Le Blanc and Rowe 1985c). In particular, it has been demonstrated (Rowe *et al* 1985b) that they very conveniently effect the suppression of states allowed by the U(d) coupling but forbidden by the Sp($d, \mathfrak{R}$)$\downarrow$U(d) branching rules. These relevant facts have their origin in the correspondence noted by Mackay (1978) between unirreps of Sp($d, \mathfrak{R}$) on the one hand, and all unirreps of $V\otimes$U(d) on the other, where V is the tangent space defined at each point of the manifold Sp($d, \mathfrak{R}$)/U(d). The unitary-Weyl algebra (cf appendix) is seen to be an explicit realisation of the tangent bundle $V\otimes$U(d) and the embedding of the sp($d, \mathfrak{R}$) algebra in the enveloping algebra of the unitary-Weyl group requires the introduction of the K^2 matrices. This embedding has been shown to encompass all the discrete series representations of the symplectic algebra (Rowe *et al* 1985a, b). We are therefore justified in calling the basis states (3.8) canonical.

The polynomials $Z^{\{\delta\}}$ can be either constructed recursively using equation (A17) or can be found in Quesne (1981). They are given by

$$Z^{\{\delta_1\delta_2\}}_{\mathrm{hw}}(\mathscr{A}) = N(x, y)\mathscr{A}^{x}_{+1}(2\mathscr{A}_{+1}\mathscr{A}_{-1} - \mathscr{A}^2_0)^y \tag{3.10}$$

where

$$N(x, y) = \frac{1}{[x!]}\left(\frac{(2x+1)!(x+y)!}{y!(2x+2y+1)!}\right)^{1/2}$$

with

$$x = (\delta_1 - \delta_2)/2$$

and

$$y = \delta_2/2.$$

4. Orthonormal basis for SU(3) ⊃ SO(3)

Introducing the spherical variables

$$\begin{aligned} g_{\alpha 0} &= g_{\alpha 3} \\ g_{\alpha\pm 1} &= \mp\frac{1}{\sqrt{2}}(g_{\alpha 1} \pm \mathrm{i} g_{\alpha 2}) \end{aligned} \tag{4.1}$$

one can easily ascertain that the Sp(2, $\mathfrak{R}$) highest weight states for the $\langle\frac{3}{2}(L)\rangle$ and $\langle\frac{3}{2}(L1)\rangle$ unirreps are given respectively by

$$\langle g|\langle\tfrac{3}{2}(L)\rangle\mathrm{hw}\rangle = \frac{1}{\sqrt{L!}}(g_{1+1})^L = \mathscr{Y}^L_L(g_1) \tag{4.2}$$

and

$$\begin{aligned} \langle g|\langle\tfrac{3}{2}(L1)\rangle\mathrm{hw}\rangle &= \frac{1}{[(L+1)(L-1)!]^{1/2}}(g_{1+1})^{L-1}\begin{vmatrix} g_{1+1} & g_{10} \\ g_{2+1} & g_{20} \end{vmatrix} \\ &= \frac{\mathrm{i}}{(L+1)^{1/2}}\mathscr{Y}^{L-1}_{L-1}(g_1)\mathscr{Y}^1_1(g_1\times g_2) \end{aligned} \tag{4.3}$$

where $\mathcal{Y}^L_M(g_\alpha)$ is a solid harmonic defined by

$$\mathcal{Y}^L_M(\boldsymbol{r}) = \left(\frac{4\pi}{(2L+1)!!}\right)^{1/2} r^L Y^L_M(\theta, \varphi).$$

One sees immediately that these states are also O(3) highest weight states belonging to unirreps $[L]$ and $[L]^*$, respectively.

By means of equations (2.6) and (2.7), we can therefore identify these $\mathrm{Sp}(2, \mathfrak{R}) \times \mathrm{O}(3)$ highest weight states with $[\mathrm{U}(3) \supset \mathrm{O}(3)] \times \mathrm{U}(2)$ highest weight states. Hence we have

$$\langle g|\{L0\}[L]M\rangle = \mathcal{Y}^L_M(g_1) \tag{4.4}$$

$$\langle g|\{L_1\}[L]M\rangle = \frac{\mathrm{i}}{(L+1)^{1/2}}[\mathcal{Y}^{L-1}(g_1) \times \mathcal{Y}^1(g_1 \times g_2)]^L_M. \tag{4.5}$$

According to equations (2.6) and (2.7), the general $\mathrm{U}(3) \supset \mathrm{O}(3)$ polynomials

$$\langle g|\{h_1 h_2\}\{\delta\}[L]L\rangle \qquad h_1 + h_2 - L \text{ even}$$

$$\langle g|\{h_1 h_2\}\{\delta\}[L]^* L\rangle \qquad h_1 + h_2 - L \text{ odd}$$

are given, respectively, by the $\mathrm{Sp}(2, \mathfrak{R}) \supset \mathrm{U}(2)$ polynomials

$$\langle g|\langle \tfrac{3}{2}(L)\rangle\{\delta\}\{\tfrac{3}{2}(h)\}\mathrm{hw}\rangle$$

$$\langle g|\langle \tfrac{3}{2}(L1)\rangle\{\delta\}\{\tfrac{3}{2}(h)\}\mathrm{hw}\rangle.$$

The construction of these $\mathrm{Sp}(2, \mathfrak{R}) \supset \mathrm{U}(2)$ states was given in the previous section. In realising them as Bargmann polynomials, one has only to couple the $\mathrm{Sp}(2, \mathfrak{R})$ highest weight polynomials (4.2)-(4.3) with the appropriate combination of lowering operators. The spherical $m = +1$ component of the $\mathrm{Sp}(2, \mathfrak{R})$ lowering operator $\mathscr{A}^{\{2\}}$ is given the normalisation

$$\mathscr{A}^{\{2\}}_{+1}(g) = \frac{1}{\sqrt{2}}(g_1 \cdot g_1). \tag{4.6}$$

5. Examples

5.1. $\langle g|\{h_1, 0\}[L]M\rangle$.

These polynomials are well known (Kramer *et al* 1981) and given by

$$\begin{aligned} \langle g|\{h_1 0\}[L]M\rangle &= \left(\frac{(2L+1)!F(h_1, L)}{L!}\right)^{1/2} (g_1 \cdot g_1)^{(h_1-L)/2} \mathcal{Y}^L_M(g_1) \\ F(h_1, L) &= \frac{[\frac{1}{2}(h_1+L)]!}{[\frac{1}{2}(h_1-L)]![h_1+L+1]!}. \end{aligned} \tag{5.1}$$

Hence

$$\langle g|\{h_1 0\}[L]L\rangle = \left(\frac{(2L+1)!F(h_1, L)}{(L!)^2}\right)^{1/2} (g_1 \cdot g_1)^{(h_1-L)/2} (g_{1+1})^L. \tag{5.2}$$

We now show how such a state is obtained with the methods of the preceding sections. Coupling the symplectic $\mathrm{Sp}(2, \mathfrak{R})$ highest weight state given by

$$\langle g|\langle\tfrac{3}{2}(L)\rangle\mathrm{hw}\rangle = \frac{1}{\sqrt{L!}}(g_{1+1})^L$$

to the polynomial $Z^{\{\delta\}}$ in $\mathscr{A}$ given by equation (3.10) we find, according to § 3,

$$\langle g|\{h_1, 0\}[L]M\rangle = K^{-1}(L, h_1)\frac{1}{\{[(h_1-L)/2]!\}^{1/2}}\left(\frac{g_1\cdot g_1}{\sqrt{2}}\right)^{(h_1-L)/2}\frac{(g_{1+1})^L}{\sqrt{L!}}. \tag{5.3}$$

Consider first the case $L=0$ and h_1 even. We find directly from equation (A16)

$$K^2(0, n+2) = (n+3)K^2(0, n). \tag{5.4}$$

Hence we obtain

$$\langle g|\{2, 0\}[0]0\rangle = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(g_1\cdot g_1)\right) = \frac{1}{\sqrt{3!}}(g_1\cdot g_1)$$

$$\langle g|\{4, 0\}[0]0\rangle = \frac{1}{(5\times 3)^{1/2}}\left(\frac{1}{2\sqrt{2}}(g_1\cdot g_1)^2\right) = \frac{1}{\sqrt{5!}}(g_1\cdot g_1)^2$$

$$\vdots$$

$$\langle g|\{h_1, 0\}[0]0\rangle = \frac{1}{[(h_1+1)!^{1/2}]}(g_1\cdot g_1)^{h_1/2}$$

in agreement with equation (5.2).

For arbitrary L, the K^2 matrices are given once more by equation (A16)

$$K^2(L, n+L+2) = (n+2L+3)K^2(L, n+L) \tag{5.5}$$

and we find

$$\begin{aligned}\langle g|\{h_1 0\}[L]L\rangle &= \left(\frac{1}{(2L+3)}\,\frac{1}{(2L+5)}\cdots\frac{1}{(h_1+L+1)}\right)^{1/2}\\ &\quad\times\left(\frac{1}{[(h_1-L)/2]!}\right)^{1/2}\left(\frac{g_1\cdot g_1}{\sqrt{2}}\right)^{(h_1-L)/2}\frac{(g_{1+1})^L}{\sqrt{L!}}\\ &= \left(\frac{(2L+1)!F(h_1, L)}{(L!)^2}\right)^{1/2}(g_1\cdot g_1)^{(h_1-L)/2}(g_{1+1})^L\end{aligned}$$

again consistent with equation (5.2).

5.2. $\langle g|\{4, 2\}\{\delta\}[2]2\rangle$.

The SU(3) unirrep $\{4, 2\}$ is the simplest example of an irrep with an SO(3) multiplicity. The branching rule is given by

$$\mathrm{SU(3)}\downarrow\mathrm{SO(3)}: \{4, 2\}\downarrow[0]+2[2]+[3]+[4] \tag{5.6}$$

i.e. there are two $L=2$ SO(3) unirreps contained in this SU(3) unirrep. Following § 3 they will be distinguished by the labels

$$\{\delta_1, \delta_2\} = \{4, 0\} \text{ and } \{2, 2\}.$$

Table 1. Indexing of $\langle\frac{3}{2}(2,0)\rangle \equiv \langle\frac{7}{2},\frac{3}{2}\rangle$ $\mathrm{Sp}(2,\Re)$ basis states.

State index	U(2) labels $\{\omega\}$	Boson labels $\{\delta\}$	Overlaps $K^2(\omega)$	
$\|1\rangle$	$\{\frac{3}{2}(2,0)\}$	$\{0,0\}$	1	
$\|2\rangle$	$\{\frac{3}{2}(4,0)\}$	$\{2,0\}$	7	
$\|3\rangle$	$\{\frac{3}{2}(3,1)\}$	$\{2,0\}$	3	
$\|4\rangle$	$\{\frac{3}{2}(2,2)\}$	$\{2,0\}$	1	
$\|5\rangle$	$\{\frac{3}{2}(6,0)\}$	$\{4,0\}$	63	
$\|6\rangle$	$\{\frac{3}{2}(5,1)\}$	$\{4,0\}$	21	
$\|7\rangle$	$\{\frac{3}{2}(4,2)\}$	$\{4,0\}$	$19/3$	$-4\sqrt{5}/3$
$\|8\rangle$	$\{\frac{3}{2}(4,2)\}$	$\{2,2\}$	$-4\sqrt{5}/3$	$44/3$

Since they will be constructed from the $\mathrm{Sp}(2,\Re)$ unirrep $\langle\frac{3}{2}(2,0)\rangle$, we first index the $\mathrm{Sp}(2,\Re)$ basis up to the desired level as shown in table 1.

The needed states are

$$|7\rangle = |\{4,2\}\{4,0\}[2]2\rangle = |\langle\tfrac{3}{2}(2,0)\rangle\{4,0\}\{\tfrac{3}{2}(4,2)\}\mathrm{hw}\rangle$$

and

$$|8\rangle = |\{4,2\}\{2,2\}[2]2\rangle - |\langle\tfrac{3}{2}(2,0)\rangle\{2,2\}\{\tfrac{3}{2}(4,2)\}\mathrm{hw}\rangle.$$

We find

$$\langle g|1\rangle = \frac{1}{\sqrt{2}}(g_{1+1})^2$$

$$Z^{\{40\}}_{+2}(g) = \frac{1}{2\sqrt{2}}(\boldsymbol{g}_1\cdot\boldsymbol{g}_1)^2$$

$$Z^{\{22\}}_{0}(g) = \frac{1}{\sqrt{6}}((\boldsymbol{g}_1\cdot\boldsymbol{g}_1)(\boldsymbol{g}_2\cdot\boldsymbol{g}_2) - (\boldsymbol{g}_1\cdot\boldsymbol{g}_2)^2)$$

and it is straightforward to verify that the states

$$[\langle g|Z^{\{40\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}}$$

and

$$[\langle g|Z^{\{22\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}}$$

have the overlaps as given in table 1.

The orthonormal basis states for the $L=2$ subspace will then be given by

$$|7\rangle = K^{-1}_{77}[\langle g|Z^{\{40\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}} + K^{-1}_{78}[\langle g|Z^{\{22\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}}$$

and

$$|8\rangle = K^{-1}_{87}[\langle g|Z^{\{40\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}} + K^{-1}_{88}[\langle g|Z^{\{22\}}|1\rangle]^{\{\frac{3}{2}(4,2)\}}$$

where

$$K^{-1}(\tfrac{3}{2}(4,2)) = \frac{1}{\lambda_+\lambda_-}\begin{pmatrix} \lambda_+\sin^2\frac{1}{2}\theta + \lambda_-\cos^2\frac{1}{2}\theta & (\lambda_+-\lambda_-)\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta \\ (\lambda_+-\lambda_-)\cos\frac{1}{2}\theta\sin\frac{1}{2}\theta & \lambda_-\sin^2\frac{1}{2}\theta + \lambda_+\cos^2\frac{1}{2}\theta \end{pmatrix}$$

with

$$1/\cos\theta = ((K^2_{77}+K^2_{88})^2 + 4(K^2_{78})^2)^{1/2}/(K^2_{77}-K^2_{88})$$

and

$$\lambda_{\pm}=[\{(K_{77}^2+K_{88}^2)\pm((K_{77}^2+K_{88}^2)^2+4(K_{78}^2)^2)^{1/2}\}/2]^{1/2}.$$

6. Discussion

In the introduction, we stated that the basis constructed above could be considered as a canonical basis for the SU(3) ⊃ SO(3) group chain. As can be seen from the algorithm for the construction of the basis and from the arguments given in § 3, this follows from a parallel claim that the basis constructed by Rowe (1984) for the $\mathrm{Sp}(d, \mathfrak{R})$ discrete series is likewise canonical when the reduction of the outer products of $\mathrm{U}(d)$ representations occurring in equation (A8) is executed in a canonical way (for example, using Biedenharn's operator patterns to solve the multiplicity problem). Thus, whereas the reduction of the outer product of $\mathrm{U}(d)$ representations is used in Littlewood's theorem to give the $\mathrm{U}(n)\downarrow\mathrm{O}(n)$ branching rules for d-rowed representations, we have shown that an explicit canonical reduction of the $\mathrm{U}(d)$ outer products defines a canonical $\mathrm{U}(n)\supset\mathrm{O}(n)$ basis for d-rowed representations.

In the subsequent articles of this series, it will be shown (Le Blanc and Rowe 1985a) that matrix elements of the SU(3) algebra can easily be calculated in the above SU(3) ⊃ SO(3) basis. We shall also give an explicit construction (Le Blanc and Rowe 1985b) of a basis of SU(3) tensor operators in our six-dimensional Bargmann space and will classify them using (modified) Biedenharn and Louck's operator patterns. This set of tensor operators can be used to define and calculate SU(3) Wigner and Racah coefficients in both Gel'fand and SU(3) ⊃ SO(3) subgroup chains. Their mode of construction will clearly indicate that the Biedenharn and Louck's resolution of the outer product of $\mathrm{U}(d)$ representations in terms of (modified) operator patterns should lend itself to easy implementation in Bargmann spaces.

Appendix

A1. Embedding of $sp(d, \mathfrak{R})$ in the enveloping algebra of the unitary-Weyl algebra $u(d)\oplus W(d(d+1)/2)$

From the coherent state theory of the symplectic group (Rowe 1984), one obtains a non-unitary realisation (with respect to the $\mathrm{U}(d)$ boson measure) of the $\mathrm{sp}(d, \mathfrak{R})$ algebra in the form

$$\begin{aligned}\Gamma(A_{ij})&=[\Lambda, a_{ij}^{\dagger}]\\ \Gamma(B_{ij})&=a_{ij}\\ \Gamma(C_{ij})&=\mathscr{C}_{ij}+(a^{\dagger}a)_{ij}\end{aligned}\tag{A1}$$

where the $a_{ij}^{\dagger}$ are $d(d+1)/2$ symmetrical Weyl (boson) operators

$$a_{ij}^{\dagger}=a_{ji}^{\dagger}$$

satisfying

$$[a_{ij}, a_{lk}^{\dagger}]=\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}\tag{A2}$$

and the $\mathscr{C}_{ij}$ are a basis for a U(d) algebra

$$[\mathscr{C}_{ij}, \mathscr{C}_{lk}] = \mathscr{C}_{ik}\delta_{jl} - \mathscr{C}_{lj}\delta_{ik} \tag{A3}$$

that commutes with the boson algebra, i.e.

$$[\mathscr{C}_{ij}, a^{\dagger}_{lk}] = [\mathscr{C}_{ij}, a_{lk}] = 0. \tag{A4}$$

Λ is a U(d) invariant operator given by

$$\Lambda = \tfrac{1}{2}\,\mathrm{Tr}[(\mathscr{C} + a^{\dagger}a)(\mathscr{C} + a^{\dagger}a)] - \tfrac{1}{4}\,\mathrm{Tr}(a^{\dagger}aa^{\dagger}a) - \tfrac{1}{4}(d+1)\,\mathrm{Tr}(a^{\dagger}a) \tag{A5}$$

where, for example,

$$\mathrm{Tr}(a^{\dagger}a) = \sum_{ij} a^{\dagger}_{ij}a_{ji}.$$

Let $V_u^{\sigma(\lambda)}$ be the carrier space for a $\mathrm{U}(d) \supset \mathrm{U}(1) \times \mathrm{SU}(d)$ unirrep $\{\sigma(\lambda)\}$ (where $\sigma = h_d$, $\lambda_i = h_i - h_d$, $i = 1, \ldots, d-1$) and let V_w be the carrier space for a representation of the Weyl (boson) algebra. The product space

$$V_{uw}^{\sigma(\lambda)} = V_u^{\sigma(\lambda)} \otimes V_w \tag{A6}$$

then carries a unirrep of the direct product unitary-Weyl algebra. Also, if $|\{\sigma(\lambda)\}\mathrm{hw}\rangle$ is the U(d) highest weight state and $|0\rangle$ is the boson vacuum, then

$$|\langle\sigma(\lambda)\rangle\mathrm{hw}) = |\{\sigma(\lambda)\}\mathrm{hw}\rangle \otimes |0\rangle \tag{A7}$$

is the highest weight state for the unitary-Weyl albegra. The boson operator $a^{\dagger}$ is a U(d) {2} tensor WRT the realisation Γ of $\mathrm{U}(d) \subset \mathrm{Sp}(d, \mathfrak{R})$ defined by (A1). Let $Z^{\{\delta\}}(a^{\dagger})$ be a normalised polynomial in the boson operators of tensor rank $\{\delta\}$ where δ is a partition with even parts. Then an orthonormal basis for $V_{uw}^{\sigma(\lambda)}$ is given by states of the form

$$|\{\sigma(\lambda)\}\{\delta\}\alpha\{\omega\}\eta) = [Z^{\{\delta\}}(a^{\dagger})|\sigma(\lambda))]_{\eta}^{\alpha\{\omega\}} \tag{A8}$$

where α is a multiplicity index and η labels a basis for the coupled U(d) unirrep $\{\omega\}$.

The U(d)-invariant operator Λ is conveniently diagonal in this basis with eigenvalues

$$\Omega(\sigma(\lambda)\delta\omega) = \tfrac{1}{4}\sum_{i=1}^{d} [2\omega_i^2 - \delta_i^2 + 2(d+1)(\omega_i - \delta_i) - 2i(2\omega_i - \delta_i)]. \tag{A9}$$

Observing that the highest weight state $|\{\sigma(\lambda)\}\mathrm{hw})$ is also a highest weight state for the realisation Γ of the $\mathrm{Sp}(d, \mathfrak{R})$ algebra, it follows that $\mathrm{Sp}(d, \mathfrak{R})$ acts irreducibly on the subspace $V_{sp}^{\sigma(\lambda)} \subset V_{uw}^{\sigma(\lambda)}$ generated from the highest weight state $|\{\sigma(\lambda)\}\mathrm{hw})$ by the $\Gamma(A)$ lowering operators. A basis for $V_{sp}^{\sigma(\lambda)}$ is obtained by eliminating from the $V_{uw}^{\sigma(\lambda)}$ basis (A8) all states for which

$$[Z^{\{\delta\}}(\Gamma(A))|\{\sigma(\lambda)\})]_{\eta}^{\alpha\{\omega\}} = 0 \tag{A10}$$

(Rowe *et al* 1985b).

A2. Sp($d, \mathfrak{R}$) reduced matrix elements

The realisation (A1) of the symplectic algebra can be made unitary with respect to the U(d) boson measure by making the transformation (Rowe 1984)

$$\gamma(X) = K^{-1}\Gamma(X)K \qquad X \in \mathrm{Sp}(d, \mathfrak{R}) \tag{A11}$$

with $K = K^{\dagger}$ Hermitian and U(d) invariant such that the action of γ is unitary. The equation $\gamma(B)^{\dagger} = \gamma(A)$, required for unitarity, then implies that K satisfies

$$K^2 a^{\dagger} K^{-2} = [\Lambda, a^{\dagger}]. \tag{A12}$$

Hence

$$K^2 a^{\dagger}_{ij} a_{ij} = [\Lambda, a^{\dagger}_{ij}] K^2 a_{ij} \tag{A13}$$

from which one derives a recursion relation for the matrix elements of K^2

$$\begin{aligned} K^2_{ij}(\sigma(\lambda); \omega) &= (i\omega | K^2 | j\omega) \\ &= (2/N(j)) \sum_{kl\omega'} \Delta\Omega(i\omega, k\omega')(k\omega' | K^2 | l\omega')(i\omega \| a^{\dagger} \| k\omega')(j\omega \| a^{\dagger} \| l\omega')^* \end{aligned} \tag{A14}$$

where

$$\Delta\Omega(i\omega, k\omega') = \Omega(i\omega) - \Omega(k\omega')$$

and where the letters i, j, k and l stand for a given $(\delta\alpha)$ multiplet,

$$N(j) = (j\omega | a^{\dagger}_{li} a_{li} | j\omega) = \sum_k n_k(j)$$

and $(i\omega \| a^{\dagger} \| k\omega)$ are reduced matrix elements of the unitary-Weyl algebra. It is easy to deduce that these are given by the product of a U(d) recoupling (Racah) coefficient times a function of (δ). Explicit expressions for Sp(2, $\mathfrak{R}$) are given below. The Sp(d, $\mathfrak{R}$) reduced matrix elements are then given by

$$\langle i\omega' \| A \| l\omega \rangle = \sum_{jk} K_{ij}(\omega')(j\omega' \| a^{\dagger} \| k\omega) K^{-1}_{kl}(\omega). \tag{A15}$$

Note that for multiplicity-free states, i.e. states $|\sigma(\lambda)\delta\alpha\omega)$ for which δ and α take only single values for the given $\sigma(\lambda)$ and ω, a recursion relation for the diagonal matrix elements of $K^2(\omega)$ is given directly from equation (A12). For multiplicity-free states we have

$$[K^2(\omega')/K^2(\omega)](i\omega' \| a^{\dagger} \| j\omega) = \Delta\Omega(i\omega', j\omega)(i\omega' \| a^{\dagger} \| j\omega). \tag{A16}$$

Specialising to Sp(2, $\mathfrak{R}$), we easily find, following Rosensteel and Rowe (1983),

$$\begin{aligned} &(\langle\sigma(\lambda)\rangle\{\delta'\}\{\omega'\} \| a^{\dagger} \| \langle\sigma(\lambda)\rangle\{\delta\}\{\omega\}) \\ &\quad = (-1)^{\omega'_1 - \omega_1} U\left(\frac{\lambda_1 - \lambda_2}{2}, \frac{\delta_1 - \delta_2}{2}, \frac{\omega'_1 - \omega'_2}{2}, 1; \frac{\omega_1 - \omega_2}{2}, \frac{\delta'_1 - \delta'_2}{2}\right)(\{\delta'\} \| a^{\dagger} \| \{\delta\}) \end{aligned}$$

where U( ) is a standard SU(2) Racah coefficient and, from Quesne (1981),

$$\begin{aligned} (\{\delta_1 + 2, \delta_2\} \| a^{\dagger} \| \{\delta_1, \delta_2\}) &= \left(\frac{(\delta_1 - \delta_2 + 2)(\delta_1 + 3)}{2(\delta_1 - \delta_2 + 3)}\right)^{1/2} \\ (\{\delta_1, \delta_2 + 2\} \| a^{\dagger} \| \{\delta_1, \delta_2\}) &= \left(\frac{(\delta_1 - \delta_2)(\delta_2 + 2)}{2(\delta_1 - \delta_2 - 1)}\right)^{1/2}. \end{aligned} \tag{A17}$$

A3. The overlap matrix

We wish to show that the square of the transformation K of § A2 is in fact the overlap matrix (3.9).

Let us denote by

$$|\Psi(\langle\sigma(\lambda)\rangle\{\delta\}\alpha\{\omega\}\eta)\rangle=[Z^{\{\delta\}}(A)|\langle\sigma(\lambda)\rangle\rangle]^{\alpha\{\omega\}}_{\eta} \tag{A18}$$

with angular brackets, the non-orthonormal basis states introduced in equation (3.4) for $d=2$ and let

$$|\Psi(\langle\sigma(\lambda)\rangle\{\delta\}\alpha\{\omega\}\eta))=[Z^{\{\delta\}}(\gamma(A))|\langle\sigma(\lambda)\rangle)]^{\alpha\{\omega\}}_{\eta} \tag{A19}$$

with rounded brackets, denote their images in the unitary-Weyl representation space. Since the isomorphism

$$A\sim\gamma(A)$$

is unitary, it follows that the overlap matrices for the two sets of states are identical, i.e.

$$\langle\Psi_{i\omega}|\Psi_{j\omega'}\rangle=(\Psi_{i\omega}|\Psi_{j\omega'}) \tag{A20}$$

where we simplify notation by putting $i=(\sigma(\lambda)\delta\alpha)$.

Since $\gamma(A)=Ka^{\dagger}K^{-1}$ and since for the highest weight $U(d)$ states

$$K|\langle\sigma(\lambda)\rangle\eta)=|\langle\sigma(\lambda)\rangle\eta)$$

it follows that

$$|\Psi(\langle\sigma(\lambda)\rangle\{\delta\}\alpha\{\omega\}\eta))=K|\langle\sigma(\lambda)\rangle\{\delta\}\alpha\{\omega\}\eta) \tag{A21}$$

where

$$|\langle\sigma(\lambda)\rangle\{\delta\}\alpha\{\omega\}\eta)=[Z^{\{\delta\}}(a^{\dagger})|\langle\sigma(\lambda)\rangle)]^{\alpha\omega}_{\eta}$$

is a member of an orthonormal basis.

It follows from (A20) and (A21) that

$$\langle\Psi_{i\omega}|\Psi_{j\omega'}\rangle=(i\omega|K^{\dagger}K|j\omega')=(i\omega|K^{2}|j\omega') \tag{A22}$$

where K is the Hermitian square root of the positive definite matrix $K^{\dagger}K$. It also follows, since

$$\langle\Psi_{i\omega}|\Psi_{j\omega'}\rangle=\delta_{\omega\omega'}\langle\Psi_{i\omega}|\Psi_{j\omega}\rangle$$

that K can be chosen diagonal in ω. Finally, an orthonormal basis of $\mathrm{Sp}(d,\mathfrak{R})$ states is given by (Rowe 1969)

$$|i\omega\rangle=\sum_{j}K^{-1}_{ij}(\sigma(\lambda);\omega)|\Psi_{j\omega}\rangle \tag{A23}$$

where $K(\sigma(\lambda);\omega)$ is the matrix with elements

$$K_{ij}(\sigma(\lambda);\omega)=(i\omega|K|j\omega).$$

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